

STABILITY OF EXPLOSIVE BOILING OF FUEL DROPLETS

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Abstract

The paper presents a study of stability of explosive boiling, where, in addition to surface evaporation, a vapor bubble grows within a highly superheated liquid droplet, immersed in a liquid or gas medium. To get better insight into the problem two simpler but related problems are studied before the full stability problem is treated. First, the stability of an evaporating highly superheated liquid droplet is analyzed, in order to estimate the influence of the outer evaporation from the droplet surface. The linear stability of the process at the final stages of explosive boiling, when the droplet forms an expanding liquid shell, is studied next. Finally, the general case of explosive boiling stability is considered. It is shown that the process is unstable as indeed has been found in existing experiments.

Introduction

The main task of an aircraft engine is to transform the chemical energy of the fuel into the kinetic energy of the air. In air-breathing jet engines, this is achieved by combustion of liquid fuel in a combustion chamber. For this chemical reaction to occur, the liquid fuel must be injected into the airstream, atomized, and the vapor must be mixed with the air. Studying the behavior of fuel droplets injected into a hot airstream is important for proper designing of combustors for air breathing jet propulsion engines<sup>1</sup>.

To achieve high efficiency, the jet engine designer generally attempts to maximize the air flow rate and the temperature in the combustion zone. When immersed in a hot combustion gas, liquid fuel droplets are suddenly and drastically heated. For high gas temperatures and high flow rates, as in modern aircraft engines, this may cause high superheating of the droplets.

The process of rapid phase transition from highly superheated liquid to vapor is called explosive boiling<sup>2-4</sup>. It is usually so fast and violent that it resembles an explosion. Explosive boiling of fuel droplets inside the combustion chamber can damage the engine.

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In this paper, we concentrate on one of the most interesting features of explosive boiling, which is a very high evaporation rate attainable in this process. One cause for such strong evaporation is thought to be the significant increase of the area of the evaporation surface caused by instability of the bubble interface<sup>2-3,5</sup>. This instability, first observed by Shepherd & Sturtevant<sup>3</sup>, manifests itself as wrinkling and roughening of the vapor bubble surface followed by its distortion. The increased area of the evaporation surface provides the necessary heat transfer to support explosive boiling.

Recently Shusser & Weihs<sup>6</sup> proposed a mathematical model describing growth of an internal vapor bubble produced by homogeneous nucleation within a liquid droplet during explosive boiling. The predictions of the model were confirmed by existing experimental results for explosive boiling of superheated droplets<sup>3,5</sup>. The instability of explosive vapor bubble growth was not, however, considered by Shusser & Weihs<sup>6</sup>. Proper understanding of this instability will throw light on the physics of explosive boiling.

Explosive boiling instability is related to the instability of laminar flames discovered by Landau<sup>7</sup> and investigated for spherical flames by Istratov & Librovich<sup>8</sup>. It is also connected to the instability of evaporation surfaces<sup>9-11</sup> and to the problem of spherical bubble stability<sup>12-15</sup>. The unperturbed state is time-dependent (growth of a spherical vapor bubble within a liquid droplet) and hence normal-mode analysis is not appropriate.

Instabilities developing on the outer surface of the liquid droplet can be capillary instability if the droplet is situated in a liquid medium or evaporating surface instability if the medium is gaseous. To analyze the interaction of the process on both interfaces one must consider two types of perturbations which we shall call "symmetric" and "antisymmetric" in analogy from stability of liquid films<sup>16</sup> or annular liquid jets<sup>17</sup>.

The physical mechanism responsible for this instability is not fully understood at present.

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Sturtevant & Shepherd<sup>18</sup> used the Landau theory to estimate stability limits and growth rates for explosive boiling. Recently Lee & Merte<sup>19</sup> showed that approximating the evaporation surface as plane and using instantaneous information for a growing spherical vapor bubble one can reasonably predict the occurrence of instability and its wavelength. Nevertheless, the understanding of rapid evaporation instability for the *spherical* case has not been achieved yet<sup>19</sup>.

Our aim is to investigate the linear stability of explosive boiling of a liquid droplet in liquid or gas medium. We concentrate on hydrodynamic aspects of the problem but consider spherical geometry and include both interfaces into the analysis. We start by studying two simpler but related problems before treating the general case.

The plan of the present paper is as follows. First, the stability of evaporation of a highly superheated liquid droplet is studied. Next, the problem of linear stability of a thin expanding liquid spherical shell is considered. Then we proceed to analyze the general case of explosive boiling stability.

#### Stability of a Highly Superheated Evaporating Liquid Droplet

Take a spherical highly superheated liquid droplet surrounded by vapor of the same composition. The droplet evaporates creating vapor flow in the host medium as shown in Fig. 1. Here no internal vapor bubbles are produced. Our purpose is to study the linear stability of this process.

The stability of evaporation has been investigated only for evaporation from plane surfaces<sup>9-11,20</sup>. Moreover, most of the work has been devoted to studying marginal stability, which may be inappropriate for investigation of the stability of evaporation (Prosperetti & Plesset<sup>11</sup>, p. 1590). Therefore, as a first stage in the study of the stability of an explosively evaporating droplet we make several simplifying assumptions based on the observations of high superheating and very strong evaporation. These assumptions are consistent with explosive boiling.

We assume that the vapor and the liquid are inviscid incompressible fluids, that evaporation rate in the base flow is constant, that the flow perturbations do not influence the evaporation rate, and that the flow field is spherically symmetric.

Let  $u_+$ ,  $p_+$  and  $u_-$ ,  $p_-$  be the velocity and the pressure on the vapor and liquid phase, respectively.  $p_0$  is the pressure far from the droplet,  $J$  is the evaporation rate (p.u. surface area),  $R=R(t)$  is the droplet radius where

$R_0 = R(0)$  and  $\rho, \rho_v$  are the densities of the liquid and the vapor, respectively;  $\alpha = \rho_v / \rho$  ( $0 < \alpha < 1$ ). Furthermore  $\sigma$  is the surface tension;  $r$  is the radial coordinate in the spherical coordinates  $(r, \theta, \varphi)$ ;  $t$  is the time.

For spherical incompressible flow with constant evaporation rate to satisfy the conservation of mass, the droplet radius must decrease linearly with time

$$R = R_0 - \frac{J}{\rho} t \quad (1)$$

From the conservation of mass and momentum at the interface, one can calculate the pressure and velocity fields (see Shusser<sup>21</sup>).

The droplet is now assumed to have the following surface shape

$$r = R(t) + \varepsilon(\theta; \varphi; t) \quad (2)$$

where  $\varepsilon \ll R$  (see insert in Fig. 1).

We denote the components of perturbation velocity in spherical coordinates and perturbation of pressure  $v_{r-}', v_{\theta-}', v_{\varphi-}', p_-'$  on the droplet and  $v_{r+}', v_{\theta+}', v_{\varphi+}', p_+'$  in the vapor. At the interface they satisfy boundary conditions which are the conservation of mass, the constancy of the evaporation rate and the conservation of three components of momentum.

Istratov & Librovich<sup>8</sup> solved a similar (but inverse) problem of spherical flame stability. Following their solution we obtain that at the unperturbed surface of the droplet  $r = R(t)$

$$v_{r-}' = \frac{\partial \varepsilon}{\partial t} \quad (3)$$

$$v_{r+}' + \varepsilon \frac{du_+}{dr} = \frac{\partial \varepsilon}{\partial t} \quad (4)$$

$$v_{\theta-}' = v_{\theta+}' + \frac{1}{R} \frac{\partial \varepsilon}{\partial \theta} u_+ \quad (5)$$

$$v_{\varphi-}' = v_{\varphi+}' + \frac{1}{R \sin \theta} \frac{\partial \varepsilon}{\partial \varphi} u_+ \quad (6)$$

$$p_- = p_+ + \varepsilon \frac{dp_+}{dr} + \sigma \left( \Lambda - \frac{2}{R} \right) \quad (7)$$

where  $\Lambda$  is the perturbed droplet surface curvature.

For inviscid fluid, the flow inside the perturbed droplet remains irrotational. Defining the perturbation potential  $\Phi_-'$  and choosing an appropriate solution of the Laplace equation

$$\Phi_-' = f(t) r^n Y_n^m(\theta; \varphi) \quad (8)$$

where  $Y_n^m$  are spherical harmonics one obtains the components of perturbation velocity and perturbation pressure

$$v_{r-}' = nfr^{n-1}Y_n^m; v_{\theta-}' = fr^{n-1}\frac{\partial Y_n^m}{\partial \theta};$$

$$v_{\phi-}' = fr^{n-1}\frac{1}{\sin \theta}\frac{\partial Y_n^m}{\partial \phi} \quad (9)$$

$$p_{-}' = -\rho \frac{df}{dt} r^n Y_n^m \quad (10)$$

The flow in the vapor phase will be rotational due to the creation of vorticity at the distorted surface of the perturbed droplet. It should be stressed that it is not possible to use the Laplace equation for pressure perturbations in the vapor phase<sup>7,8</sup> because there is a vapor flow in the unperturbed case. It turns out that our problem (outer evaporation from the droplet surface) is much more difficult than the problem of Istratov & Librovich<sup>8</sup> (expansion of spherical flame) or the problem of evaporation into an inner bubble.

Due to the velocity perturbation  $\vec{v}_+$  being solenoidal, we can divide it into toroidal and poloidal parts  $\vec{T}, \vec{S}$  (Chandrasekhar<sup>22</sup>, p. 225).

The vorticity  $\vec{\Omega}$  can also be divided

$$\vec{\Omega} = \vec{\tilde{T}} + \vec{\tilde{S}} \quad (11)$$

where the functions  $\tilde{T}, \tilde{S}$  satisfy (Ref. 22, p. 226)

$$\tilde{T} = \frac{n(n+1)}{r^2} S - \frac{d^2 S}{dr^2} \quad (12)$$

$$\tilde{S} = T \quad (13)$$

The radial component of vorticity is

$$\Omega_r = \frac{n(n+1)}{r^2} T(r) Y_n^m \quad (14)$$

We now show that in the linear approximation  $\Omega_r$  vanishes.

Indeed, linearizing the vorticity equation for inviscid flow we obtain for  $\Omega_r$

$$\frac{\partial \Omega_r}{\partial t} + u_+ \frac{\partial \Omega_r}{\partial r} - \Omega_r \frac{du_+}{dr} = 0 \quad (15)$$

and therefore  $\Omega_r$  is not identically zero only if it is created at the perturbed interface (2). On the other hand, the vorticity is of order  $O(\varepsilon)$  and therefore neglecting second-order terms one can calculate it at the surface of the unperturbed droplet  $r = R$ . Then using the definition of vorticity and boundary conditions (5-6) one obtains that the radial component of vorticity is

continuous at the interface. It is zero inside the droplet, so that it must vanish in the vapor too.

We have therefore obtained that  $\Omega_r$  is a second-order quantity and hence negligible in the investigation of linear stability. Thus we put

$$T(r) \equiv 0 \quad (16)$$

That is, in the vapor phase the flow-field has only a poloidal part and the vorticity field has only a toroidal part. This is in accordance with Prosperetti (Ref. 23, p. 344) who stated that in perturbed spherical flows a poloidal part of the vorticity cannot be generated if it vanishes at the initial moment.

Returning to the calculation of the velocity field within the vapor, relation (12) is now an ordinary differential equation, with general solution

$$S = \frac{C(t)}{r^n} + D(t)r^{n-1} + S_p \quad (17)$$

where  $S_p$  is the particular solution.

We assume that when  $r \rightarrow \infty$  the vorticity tends to zero sufficiently rapidly so that  $S_p \rightarrow 0$  too. Then

$$D(t) \equiv 0 \quad (18)$$

We are interested in the behavior of  $S(r; t)$  near the droplet surface. Therefore we approximate  $\tilde{T}(r; t)$  by its value at the droplet surface, which we denote  $F(t)$

$$\tilde{T}(r; t) \approx \tilde{T}(R(t); t) \equiv F(t) \quad (19)$$

Then calculating  $S_p$  we obtain

$$S = \frac{C(t)}{r^n} + \frac{F(t)r^2}{n(n+1)-2} \quad (20)$$

The solution (20) always exists, as we are interested in the perturbation modes for which  $n \geq 2$ .

From (16) and (20) one obtains the perturbed velocity field in the vapor, while the pressure perturbation can be calculated from the linearized Euler equation<sup>21</sup>. In addition, the droplet surface distortion is

$$\varepsilon = a_1(t) Y_n^m(\theta; \varphi), \quad n \geq 2 \quad (21)$$

Substituting the pressure and velocity fields into the boundary conditions (3-7), we can obtain the equation for the amplitude of the droplet surface perturbation  $a_1(t)$  (see Shusser<sup>21</sup>). It follows from the solution of this equation that when  $R \rightarrow 0$

$$\frac{a_1}{R} \sim R^{\frac{2\beta-1}{4}} \quad (22)$$

Here

$$\beta = \frac{(n-1)}{1-\alpha} \left( 3 - \frac{1}{n+1} + \alpha \right) \quad (23)$$

One sees that  $\beta > 8/3$  when

$n \geq 2$  and  $0 < \alpha < 1$  and therefore

$a_1 / R \rightarrow 0$  when  $R \rightarrow 0$ , i.e., droplet

surface perturbations tend to zero faster than the droplet radius. The solution is thus stable.

A question arises about the validity of this analysis, due to the assumption of  $\frac{\varepsilon}{R} \ll 1$

(equation 2) which is only justified *a posteriori* here.

However, returning to our main objective, stability of explosive boiling, the results of this section enable us to ignore the outer evaporation from the droplet surface when analyzing the stability of a liquid droplet boiling explosively, as  $R$  never tends to zero in the full problem, where an internal vapor bubble is produced.

The decay is faster for higher wave numbers and when vapor and liquid densities are closer in value ( $\alpha \rightarrow 1$ ), as seen from (23).

It is interesting to compare our results for evaporating droplets and the results for expanding spherical flames<sup>8</sup> with those for spherical bubbles<sup>12-15</sup>. In the latter case there is no flow through the bubble surface and therefore the perturbed flow remains irrotational.

One sees from the comparison that the results are opposite. The growing bubble ( $R \rightarrow \infty$ ) is stable and the collapsing bubble ( $R \rightarrow 0$ ) is unstable. On the other hand, the expanding flame ( $R \rightarrow \infty$ ) is unstable and evaporating droplets ( $R \rightarrow 0$ ), as we have just shown, are stable.

#### Stability of a Thin Expanding Spherical Shell

Consider a thin spherical liquid shell of density  $\rho$  and surface tension  $\sigma$  expanding in a gas medium as shown in Fig. 2. The shell is characterized by its mean geometric radius  $R(t)$  and its thickness  $h(t)$ . This model describes the late stage of explosive boiling.

The instability of an expanding liquid shell is stronger for boiling in a gas. This results from the large difference in density between liquid and gas, which facilitates the motions of shell segments. Therefore we analyze this case.

We assume that the shell is thin relative to its radius ( $h \ll R$ ), that the unperturbed (base) solution is spherically symmetric, that the liquid is inviscid and incompressible, that gas and internal vapor density are negligible, that evaporation from the outer surface of the droplet is negligible, and that the pressure in the host gas

$p_\infty$  and the inner bubble  $p_i$  is constant and uniform.

The last assumption corresponds to the existence of very weak evaporation into the inner bubble. Because it is weak one can neglect its influence on the shell mass and the stability, but due to the low density of the vapor it is sufficient to support constant pressure within the bubble.

We utilize the assumption of a thin shell by neglecting quantities of order  $h^2 / R^2$  both in the base solution and in the perturbed flow.

Let  $4\pi M$  be the mass of the shell. Then from the conservation of mass

$$4\pi M = 4\pi h R^2 \rho \left( 1 + \frac{h^2}{12R^2} \right) \quad (24)$$

and after neglecting the second-order quantities

$$hR^2 = \frac{M}{\rho} = \text{const} \quad (25)$$

To this approximation the flow-field in the shell is (the dot denotes a time derivative)

$$v = \frac{R^2}{r^2} \dot{R} \quad (26)$$

Then from the unsteady Bernoulli equation of hydrodynamics<sup>24</sup> and the boundary conditions at both surfaces one obtains in the linear approximation the equation for the shell radius

$$\ddot{R} + \frac{4\sigma}{M} R - \frac{(p_i - p_\infty)}{M} R^2 = 0 \quad (27)$$

This equation does not include a  $\dot{R}$  term. This means that there is no damping term in the equation and therefore thin shell oscillations do not decay or amplify in the linear approximation.

Integrating (27), we obtain for the expanding shell

$$\dot{R} = \sqrt{\frac{2}{3} b R^3 - a R^2 + c} \quad (28)$$

where

$$a = \frac{4\sigma}{M} ; \quad b = \frac{p_i - p_\infty}{M} \quad (29)$$

and  $c$  is defined by initial conditions.

The expression within the radical in (28) must be positive. This can be considered as a condition on the shell radius  $R$  for sustaining a thin shell.

We perturb the base solution so the shell radius and the shell thickness are now  $R(t) + \eta(t; \theta; \varphi)$  and  $h(t) + \xi(t; \theta; \varphi)$ . The perturbations are small and therefore  $\eta \ll R$  and  $\xi \ll h$  and the terms of order  $O(\eta^2)$  and  $O(\xi^2)$  are negligible.

The perturbed shell is situated between two slightly perturbed spheres. Their equations are

$$r = R \mp \frac{h}{2} + \eta \mp \frac{\xi}{2} \quad (30)$$

From conservation of mass

$$4\pi M = \rho(V_2 - V_1) \quad (31)$$

where  $V_1$  and  $V_2$  are the volumes of the inner and outer perturbed spheres, respectively. To linear approximation

$$V_1 = \frac{4}{3}\pi \left(R - \frac{h}{2}\right)^3 + \left(R - \frac{h}{2}\right)^2 \int_0^{2\pi} \int_0^\pi \left(\eta - \frac{\xi}{2}\right) d\theta d\varphi \quad (32)$$

$$V_2 = \frac{4}{3}\pi \left(R + \frac{h}{2}\right)^3 + \left(R + \frac{h}{2}\right)^2 \int_0^{2\pi} \int_0^\pi \left(\eta + \frac{\xi}{2}\right) d\theta d\varphi \quad (33)$$

Substituting (25), (32-33) into (31) and neglecting the second-order terms in  $h$  we obtain

$$\int_0^{2\pi} \int_0^\pi (R^2 \xi + 2Rh\eta) d\theta d\varphi = 0 \quad (34)$$

One can conclude that there exist two types of perturbations (see Fig. 2). If the perturbation of the radius  $\eta$  is not identically zero then

$$\xi = -\frac{2h}{R}\eta \quad (35)$$

The perturbations of the shell radius are analogous to the antisymmetric (sinuous) perturbations observed in plane liquid films<sup>16</sup> or annular liquid jets<sup>17</sup>. For spherical flows, they also cause perturbations of the shell thickness, given by (35).

It is possible to perturb the thickness leaving the shell radius unchanged provided the shell volume does not change, i.e., if

$$\int_0^{2\pi} \int_0^\pi \xi d\theta d\varphi = 0 \quad (36)$$

This type of perturbation is analogous to the symmetric (varicose) perturbations for the plane film<sup>16</sup>. Any perturbation can be represented as a linear combination of symmetric and antisymmetric ones. Therefore each type of perturbation can be analyzed separately. We begin the analysis for the antisymmetric case.

The perturbation potential is a solution of the Laplace equation and therefore the full potential of the perturbed flow  $\Phi$  is given by

$$\Phi = -\frac{R^2 \dot{R}}{r} + \left(\frac{a_1(t)}{r^{n+1}} + a_2(t)r^n\right) Y_n^m(\theta; \varphi) \quad (37)$$

Assuming for the perturbation  $\eta$  a solution of the form

$$\eta = a_3(t) Y_n^m(\theta; \varphi) \quad (38)$$

we can find the functions  $a_1(t), a_2(t), a_3(t)$  from the boundary conditions, which are two kinematic boundary conditions (Lamb<sup>24</sup>, p. 7) and the conservation of momentum<sup>21</sup>.

We are interested in the behavior of the solution for  $t \rightarrow \infty$ , i.e.  $R \rightarrow \infty$ . Neglecting the terms of order  $R^{-2}$ , one obtains (see Shusser<sup>21</sup>) that when  $R \rightarrow \infty$

$$\frac{a_3}{R} \sim \frac{C_1}{R^3} + C_2 R^{1/2} \quad (39)$$

For stability  $a_3/R$  must remain bounded. Therefore one of the solutions is always unstable and the conclusion is that the expansion of a thin liquid shell is unstable for antisymmetric perturbations. It can be noted that the instability is relatively weak, growing as  $R^{1/2}$ .

Considering symmetric perturbations, we first mention that they are perturbations of the shell thickness. They are initially much smaller than the thickness itself even in the thin-shell approximation. Therefore, in analyzing stability of symmetric perturbations we retain the terms of order  $O(h^2/R^2)$  in the base solution.

Repeating the previous calculations including the  $O(h^2/R^2)$  terms, we obtain the shell thickness  $h$  and the flow potential  $\Phi$  are now

$$h = \frac{M}{\rho R^2} \left(1 - \frac{M^2}{12\rho^2 R^6}\right) \quad (40)$$

$$\Phi = -\frac{R^2}{r} \dot{R} \left(1 - \frac{3h^2}{4R^2}\right) \quad (41)$$

Proceeding as previously, we look for a solution for the thickness perturbation  $\xi$  of the form

$$\xi = a_3(t) Y_n^m(\theta; \varphi) \quad (42)$$

with the potential in the perturbed flow

$$\Phi = -\frac{R^2}{r} \dot{R} \left(1 - \frac{3h^2}{4R^2}\right) + \left(\frac{a_1(t)}{r^{n+1}} + a_2(t)r^n\right) Y_n^m(\theta; \varphi) \quad (43)$$

Then the equation for  $a_3(t)$  is

$$\frac{d^2 a_3}{dR^2} + P(R) \frac{da_3}{dR} + Q(R) a_3 = 0 \quad (44)$$

where  $P(R)$  and  $Q(R)$  are given by Shusser<sup>21</sup>.

From the leading terms of  $P(R), Q(R)$  for  $R$  sufficiently large

$$Q > 0 ; \quad \frac{dQ}{dR} + 2PQ < 0 \quad (45)$$

and therefore (Birkhoff<sup>13</sup>) each solution of (44) tends to infinity when  $R \rightarrow \infty$ .

To find the behavior of the solution for  $R \rightarrow \infty$ , we substitute in Eq. (44) the first terms in the asymptotic expansions of  $P$  and  $Q$  for large  $R$ . Comparing the behavior of the shell thickness  $h$  and the amplitude of its perturbation  $a_3$ , one sees that when  $R \rightarrow \infty$  and  $h \rightarrow 0$

$$\frac{a_3}{h} \sim h^{-13/8} \xrightarrow{h \rightarrow 0} \infty \quad (46)$$

That is, there is strong instability for all wave numbers.

We obtained that both perturbations of the shell radius (antisymmetric) and perturbations of the shell thickness (symmetric) are unstable. When the shell radius  $R$  tends to infinity and the shell thickness  $h$  tends to zero, the appropriate perturbations normalized by  $R$  or  $h$ , respectively, are asymptotic to  $\sqrt{R}$  and  $h^{-13/8}$ .

Squire<sup>16</sup> has shown that for thin film stability both types of perturbations are unstable. In variance with our solution, however, in the plane case antisymmetric perturbations grow much faster than symmetric ones. For a spherical shell we obtained weak instability for perturbations of the shell radius and strong instability for those of the thickness.

One sees that the fact that both shell radius and shell thickness are time-dependent in our spherical case does not change the general conclusion about stability but changes its nature because the most dangerous perturbations are now of different type. The weak instability obtained for the antisymmetric perturbations may tie in with the related process of gas bubble growth with a constant pressure inside the bubble, which is stable<sup>12-15</sup>.

The goal of this section was to identify the most dangerous perturbations for stability of explosive boiling. These are the symmetric (varicose) perturbations.

#### Stability of Explosive Boiling

The physical situation is depicted in Fig. 3. A vapor bubble of radius  $R_1(t)$  grows within a highly superheated liquid droplet of radius  $R_2(t)$ , which is situated in liquid or gas medium. For simplicity, we assume that the bubble is situated in the center of the droplet. Though an approximation, this assumption was shown reasonable for a broad range of physical situations<sup>6</sup>.

We assume that all the fluids are inviscid and incompressible and the evaporation rate is

constant and is not influenced by the perturbations. We also neglect the flow inside the vapor bubble. Utilizing the results of the previous sections we neglect the effects of possible evaporation from the outer surface of the droplet and consider only symmetric perturbations.

Due to the constant rate of evaporation, the rate of growth of the bubble radius will also be constant<sup>3</sup>. Denoting this constant rate  $U$  we write

$$R_1 = Ut \quad (47)$$

and using conservation of mass

$$\frac{R_2}{R_0} = \left[ 1 + (1 - \alpha) \left( \frac{Ut}{R_0} \right)^3 \right]^{1/3} \quad (48)$$

Here  $R_0$  is the initial radius of the droplet;  $\alpha = \rho_v / \rho$ ;  $\rho, \rho_v$  are the densities of the droplet liquid and the vapor, respectively.

We add small symmetric perturbations  $\varepsilon(\theta; \varphi; t)$  ( $\varepsilon \ll R_1, R_2$ ) to both interfaces so their equations are now

$$r_1 = R_1(t) + \varepsilon(\theta; \varphi; t) \quad (49)$$

$$r_2 = R_2(t) - \varepsilon(\theta; \varphi; t) \quad (50)$$

From conservation of mass

$$\int_0^{2\pi} \int_0^\pi \varepsilon(\theta; \varphi; t) \sin \theta d\theta d\varphi = 0 \quad (51)$$

Distortion of the droplet shape causes perturbations of the velocity  $v_{r_1}, v_{\theta_1}, v_{\varphi_1}$  and the pressure  $p_i$  ( $i=1,2,3$  for the vapor bubble, the liquid droplet and the host fluid, respectively). These perturbations satisfy the boundary conditions at the surface of the bubble and the surface of the droplet.

We write the conditions of conservation of mass, conservation of the tangential components of momentum and constancy of the evaporation rate at the surface of the unperturbed bubble  $r = R_1$  as

$$v_{r_1}' = \frac{\partial \varepsilon}{\partial t} \quad (52)$$

$$v_{r_2}' + \varepsilon \frac{\partial v_{r_2}'}{\partial r} = \frac{\partial \varepsilon}{\partial t} \quad (53)$$

$$v_{\theta_1}' = v_{\theta_2}' + \frac{1}{R_1} \frac{\partial \varepsilon}{\partial \theta} v_{r_2}' \quad (54)$$

$$v_{\varphi_1}' = v_{\varphi_2}' + \frac{1}{R_1 \sin \theta} \frac{\partial \varepsilon}{\partial \varphi} v_{r_2}' \quad (55)$$

The kinematic conditions at the surface of the unperturbed droplet  $r = R_2$  are

$$v_{r_2}' + \frac{\partial \varepsilon}{\partial t} = \varepsilon \frac{\partial v_{r_2}'}{\partial r} \quad (56)$$

$$v_{r_3}' + \frac{\partial \varepsilon}{\partial t} = \varepsilon \frac{\partial V_{r_3}}{\partial r} \quad (57)$$

Conservation of the normal component of momentum at  $r = R_1$  and at  $r = R_2$  results in the dynamic boundary condition

$$\begin{aligned} & \left( p_3' - \varepsilon \frac{\partial p_3}{\partial r} \right) \Big|_{r=R_2} - p_1' \Big|_{r=R_1} + \\ & \sigma \left( \Lambda_1 + \Lambda_2 - \frac{2}{R_1} - \frac{2}{R_2} \right) = \\ & = \left( p_2' - \varepsilon \frac{\partial p_2}{\partial r} \right) \Big|_{r=R_2} - \left( p_2' + \varepsilon \frac{\partial p_2}{\partial r} \right) \Big|_{r=R_1} \quad (58) \end{aligned}$$

Here  $V_{r_2}, V_{r_3}, P_2, P_3$  denote the unperturbed solution.

We write the perturbation  $\varepsilon$  as, using (51)

$$\varepsilon(\theta; \varphi; t) = f_4(t) Y_n^m(\theta; \varphi) \quad (n \neq 0) \quad (59)$$

The perturbed pressure and velocity fields in the vapor bubble, the liquid droplet and the host liquid were calculated by Shusser<sup>21</sup>. By substituting them into the boundary conditions (52-58) the following equation for the amplitude of the shape perturbation is obtained

$$A(t) \frac{d^2 f_4}{dt^2} + B(t) \frac{df_4}{dt} + C(t) f_4 = 0 \quad (60)$$

The functions  $A(t)$ ,  $B(t)$ ,  $C(t)$  are written out in Ref. 21.

We start the analysis by defining the conditions for stability. The process of explosive boiling occurs during a finite time until the droplet evaporates completely, i.e., until

$$t = t_f = \frac{R_0}{U\alpha^{1/3}} \quad (61)$$

We propose to analyze the stability by investigating the behavior of the solution of (60) when  $t \rightarrow t_f$ . It is possible that the perturbations grow so fast that the droplet is destroyed before  $t_f$ . Therefore our method can prove only instability of the process.

Taking new variables

$$\tau = t_f - t; \quad x = \frac{U\alpha^{1/3}}{R_0} \tau; \quad y = \frac{\alpha^{1/3}}{R_0} f_4 \quad (62)$$

one can write the approximate form of (60) in the limit  $\tau \rightarrow 0$  as

$$\begin{aligned} & x(a_1 x + a_2) \frac{d^2 y}{dx^2} + (b_1 x + b_2) \frac{dy}{dx} + \\ & (c_1 x + c_2) y = 0 \end{aligned} \quad (63)$$

The coefficients  $a_1, a_2, b_1, b_2, c_1, c_2$  are given by Shusser<sup>21</sup>.

No general analytic solutions for Eq. (60) are known. Studying the behavior of the solution when  $x \rightarrow 0$  by using the method of Fröbenius<sup>25</sup>, one can show that in physical variables the leading behavior is

$$\frac{\varepsilon}{R_2 - R_1} \sim (t_f - t)^{\frac{2(n+1)}{\beta n - \alpha(n+1)}} \quad (64)$$

$$\frac{\varepsilon}{R_2 - R_1} \sim \frac{1}{(t_f - t)} \quad (65)$$

Here  $\beta = \rho_1/\rho$ .

Thus, there exist two solutions (64-65) for symmetric perturbations of the process of explosive boiling of a liquid droplet. The stability of the former depends on the ratio of the densities of the vapor and the host fluid  $\alpha/\beta$ . For low density of the host fluid ( $\alpha > \beta$ ) the solution (64) is also unstable and grows faster than (65). When  $\beta > \alpha$  only low wave numbers are unstable in (64). For high density of the host liquid, the solution (64) is stable. That means that the droplet breakup is easier in the gas medium than in the liquid, which is a physically reasonable result.

On the other hand, the solution (65) is always unstable. One can therefore conclude that the process of the explosive boiling of a liquid droplet is unstable.

### Conclusions

In this paper we analyzed the stability of explosive boiling of a liquid droplet. Two related problems were considered. We showed that evaporation of a highly superheated droplet is stable and we studied the stability of a thin expanding liquid shell. The results of the former problem can be also used for a contracting spherical flame while investigation of the latter may throw some light on the process of droplet breakup at the final stages of the boiling.

The results of these related problems were used to justify the assumptions necessary in studying the general case of explosive boiling stability. It was found that the process is unstable as indeed was observed in the experiments of Shepherd & Sturtevant<sup>3</sup>. The instability was obtained for all wave numbers, which again is consistent with the observation of Shepherd & Sturtevant (Ref. 3, p. 388) that the roughening of the bubble surface occurs on many length scales.

### Acknowledgements

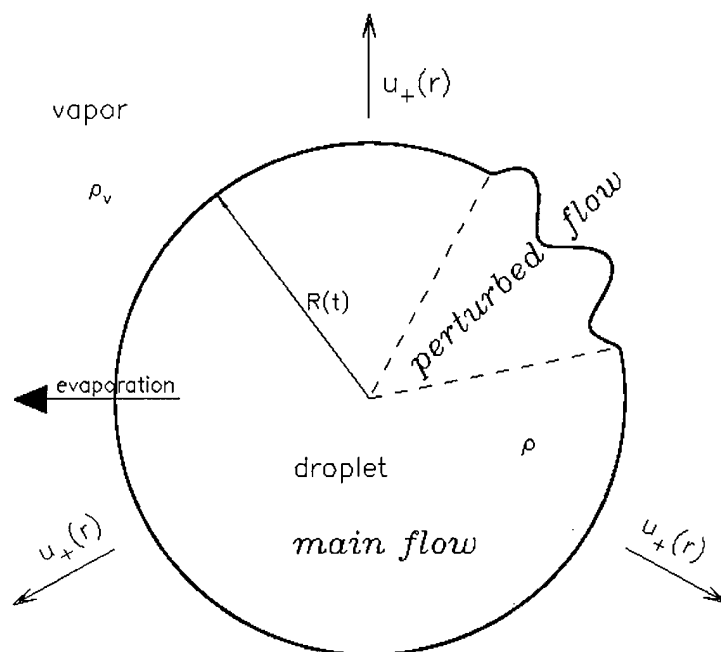
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Fig. 1. A highly superheated evaporating droplet.

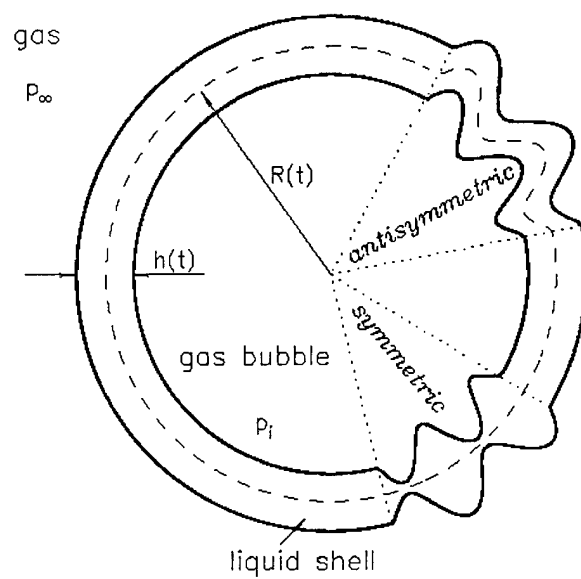


Fig. 2. A thin expanding liquid shell.

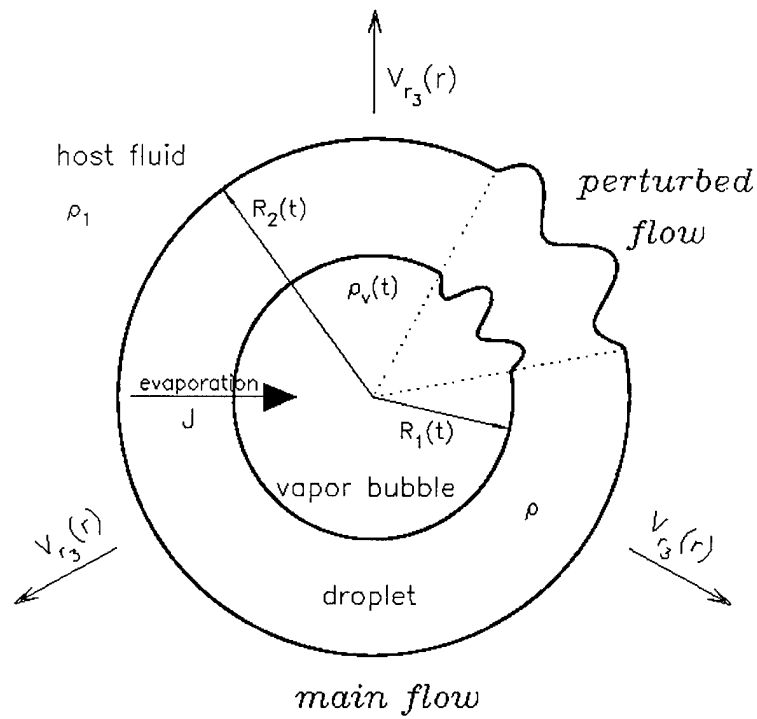


Fig. 3. Explosive boiling of a liquid droplet.